Lecture 14: October 21

The general problem. In order to prove Theorem 10.3, we need to solve the following problem in linear algebra. Suppose that V is a finite-dimensional complex vector space, with a representation by $\mathfrak{sl}_2(\mathbb{C})$. We denote by $H, X, Y \in \operatorname{End}(V)$ the images of the standard generators of $\mathfrak{sl}_2(\mathbb{C})$. Suppose that $h \colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ is a hermitian pairing compatible with the $\mathfrak{sl}_2(\mathbb{C})$-action; concretely, this means that

$$h(Yv', v'') = h(v', Yv''), \quad h(Xv', v'') = h(v', Xv''), \quad h(Hv', v'') = -h(v', Hv'')$$

for every $v', v'' \in V$. We also suppose that V has an increasing filtration $F = F^\bullet V$, with the following three properties:

(1) For every $p \in \mathbb{Z}$, one has $H(F^p) \subseteq F^p$.
(2) For every $p \in \mathbb{Z}$, one has $Y(F^p) \subseteq F^{p-1}$.
(3) The filtration $e^{-\frac{1}{2}Y} F$ is the Hodge filtration of a Hodge structure of weight n, polarized by h.

For lack of a better term, let us call this data a *polarized* $\mathfrak{sl}_2(\mathbb{C})$*-Hodge structure of weight* n. This expression is nedeed only temporarily, because we are going to deduce from these conditions that V is actually a Hodge-Lefschetz structure of central weight n (as defined in Lecture 3). Here is the precise statement.

Theorem 14.1. *With notation and assumptions as above,*

$$V = \bigoplus_{k \in \mathbb{Z}} E_k(H)$$

is a Hodge-Lefschetz structure of central weight n, which is polarized by h. More precisely, each eigenspace $E_k(H)$ has a Hodge structure of weight $n+k$, and

$$F^p V = \bigoplus_{k \in \mathbb{Z}} F^p E_k(H).$$

Moreover, if a semisimple endomorphism $S \in \operatorname{End}(V)$ commutes with H and Y, is compatible with h, and satisfies $S(F^p) \subseteq F^p$ for every $p \in \mathbb{Z}$, then S is an endomorphism of the Hodge-Lefschetz structure.

Concretely, this is saying the following things:

(1) Each eigenspace $E_k(H)$ has a Hodge structure of weight $n+k$, whose Hodge filtration is given by intersecting F with the subspace $E_k(H)$.
(2) For each $k \in \mathbb{Z}$, the mapping $Y \colon E_k(H) \to E_{k-2}(H)(-1)$ is a morphism of Hodge structures (of weight k).
(3) For each $k \geq 0$, the Hodge structure on the primitive subspace
$$E_k(H) \cap \ker X = \ker\Bigl(Y^{k+1} \colon E_k(H) \to E_{-k-2}(H)\Bigr)$$
is polarized by the pairing $(v', v'') \mapsto (-1)^k h(v', Y^k v'')$.

Example 14.2. This result implies Theorem 10.3, by setting $Y = R_N$ and $S = R_S$.

Here is another interpretation, closer to the language that Schmid is using in his paper. Recall from Lecture 9 that the nilpotent endomorphism $Y \in \operatorname{End}(V)$ has an associated monodromy weight filtration $W = W_\bullet(Y)$, with the property that $Y(W_k) \subseteq W_{k-2}$. By construction,

$$\operatorname{gr}_k^W = W_k / W_{k-1} \cong E_k(H),$$

and so this has a Hodge structure of weight $n+k$, whose Hodge filtration is induced by F. After a shift in indexing, this is saying that the increasing weight filtration $W_{\bullet - n}$, together with the decreasing Hodge filtration $F^\bullet$, are part of a *mixed Hodge structure* on V. Schmid summarizes the results about Y and h by saying that the mixed Hodge structure is "polarized by the pairing h and the nilpotent operator Y". This version has the advantage of not mentioning the (auxiliary) splitting H.

A tedious example. Let us start by analyzing a small example by hand. This is going to be quite tedious, but it will hopefully help you appreciate the actual proof of Theorem 14.1.

Example 14.3. Suppose that $\dim V = \mathbb{C}a \oplus \mathbb{C}b \oplus \mathbb{C}c$, with $\mathfrak{sl}_2(\mathbb{C})$-action given by

$$Ha = 2a, \quad Hb = 0, \quad Hc = -2c, \quad Ya = b, \quad Yb = 2c.$$

This is the 3-dimensional irreducible representation, with $b = Ya$ and $c = \frac{1}{2}Y^2 a$ (to simplify the formulas). We have

$$h(b,b) = h(Ya, Ya) = h(a, Y^2 a) = 2h(a,c),$$

and so $h(a,c) = h(c,a)$ is real, and $h(b,b) = 2h(a,c)$; on all the remaining basis vectors, the pairing is zero. In particular, the signature of the pairing (on $\mathbb{R}^6$) is either $(4,2)$ or $(2,4)$, depending on whether $h(a,c)$ is positive or negative.

We now assume that we are given a filtration F such that $e^{-\frac{1}{2}Y}F$ defines a Hodge structure of weight $n = 2$, polarized by h; since we can always shift the indices up or down, we may assume without loss of generality that $c \in F^0$ but $c \notin F^1$. Let

$$V = \bigoplus_{p+q=2} V^{p,q}$$

be the Hodge decomposition. What can we deduce about F?

(1) We must have $F^3 = 0$. Indeed, if $xa + yb + zc \in F^3$, then

$$2xc = Y^2(xa + yb + zc) \in F^1,$$

and so $x = 0$; but then $2yc = Y(yb + zc) \in F^2$, hence also $y = z = 0$. For the same reason, $\dim F^2 \leq 1$: indeed, $xa + yb + zc \in F^2$ can only be nonzero if $x \neq 0$. Thus we see that

$$c = c_0 + c_1 + c_2,$$

with $c_p \in V^{p,q}$; since $c = e^{-\frac{1}{2}Y}c \notin e^{-\frac{1}{2}Y}F^1$, we must have $c_0 \neq 0$.

(2) It is also easy to see that c_2 spans $V^{2,0} = e^{-\frac{1}{2}Y}F^2$. As $\dim F^2 \leq 1$, this amounts to saying that $c_2 = 0$ implies $V^{2,0} = 0$. Let $v = xa+yb+zc \in V^{2,0}$ be an arbitrary vector. Then

$$h(v, c_2) = h(xa + yb + zc, c) = x \cdot h(a,c),$$

and so $c_2 = 0$ implies $x = 0$, hence $v = 0$. Therefore $V^{2,0} = \mathbb{C}c_2$.

(3) Since the Hodge decomposition is orthogonal with respect to h, we get

$$0 = h(c,c) = h(c_0,c_0) + h(c_1,c_1) + h(c_2,c_2).$$

Now $h(c_0,c_0) > 0$, whereas $h(c_1,c_1) \leq 0$ and $h(c_2,c_2) \geq 0$. This can only happen if $c_1 \neq 0$. Let $u, v, w \in \mathbb{C}$ be such that $c_1 = ua + vb + wc$. Then

$$e^{\frac{1}{2}Y}c_1 = ua + \left(v + \frac{u}{2}\right)b + \left(w + v + \frac{u}{4}\right)c \in F^1,$$

and since F is compatible with the decomposition into H-eigenspaces, all three summands belong to F^1. In particular, the third summand has to vanish (because $c \notin F^1$), and so $w = -\frac{1}{4}u - v$. Since

$$0 > h(c_1,c_1) = h(c_1,c) = u \cdot h(a,c)$$

we also get $u \neq 0$, hence $a \in F^1$. But then $b = Ya \in F^0$, and so we actually have $F^0 = V$. In terms of the Hodge decomposition, this means that

$$V = V^{0,2} \oplus V^{1,1} \oplus V^{2,0}.$$

(4) Now we argue that $c_2 \neq 0$. Suppose to the contrary that $c_2 = 0$; then also $F^2 = 0$. Therefore $V^{1,1} = e^{-\frac{1}{2}Y} F^1$, and in particular

$$e^{-\frac{1}{2}Y} a = a - \frac{1}{2} b + \frac{1}{4} c \in V^{1,1}.$$

Since h is a polarization, we get

$$h(a,c) = h\left(a, e^{-Y} a\right) = h\left(e^{-\frac{1}{2}Y} a, e^{-\frac{1}{2}Y} a\right) < 0.$$

The signature of h is therefore $(2,4)$, and so $\dim V^{1,1} = 2$ and $\dim V^{0,2} = 1$. In particular, $\dim F^1 = 2$, and because F is compatible with the decomposition into H-eigenspaces (and $c \notin F^1$), we get $F^1 = \mathbb{C}a \oplus \mathbb{C}b$, and therefore

$$V^{1,1} = \mathbb{C}e^{-\frac{1}{2}Y} a \oplus \mathbb{C}e^{-\frac{1}{2}Y} b.$$

Recall that $c_1 = ua + vb + wc$, where $w + v + \frac{1}{4}u = 0$. Since the Hodge decomposition is orthogonal with respect to h, we get

$$h\left(c_1, e^{-\frac{1}{2}Y} a\right) = h\left(c, e^{-\frac{1}{2}Y} a\right) = h(a,c)$$
$$h\left(c_1, e^{-\frac{1}{2}Y} b\right) = h\left(c, e^{-\frac{1}{2}Y} b\right) = 0.$$

It follows that $w - v + \frac{1}{4}u = 1$, hence $v = -\frac{1}{2}$; also $u = 2v = -1$, and so

$$c_1 = -a - \frac{1}{2} b + \frac{3}{4} c.$$

But then $h(c_1, c_1) = h(c_1, c) = -h(a,c) > 0$, which contradicts $c_1 \in V^{1,1}$. The conclusion is that $c_2 \neq 0$ after all.

(5) We have $e^{\frac{1}{2}Y} c_2 \in F^2$, and since $b, c \notin F^2$, we must have $a \in F^2$ and $e^{\frac{1}{2}Y} c_2 = xa$ for some $x \in \mathbb{C}$. Then $c_2 = x \cdot e^{-\frac{1}{2}Y} a$, and from

$$|x|^2 h(a,c) = |x|^2 h(a, e^{-Y} a) = h(c_2, c_2) = h(c_2, c) = x \cdot h(a,c),$$

we deduce that $x = 1$ and that $h(a,c) > 0$. Thus $c_2 = e^{-\frac{1}{2}Y} a = a - \frac{1}{2}b + \frac{1}{4}c$. Recall that $c_1 = ua + vb + wc$, with $w + v + \frac{1}{4}u = 0$. This time, we have

$$0 = h(c_1, c_2) = h\left(c_1, e^{-\frac{1}{2}Y} a\right),$$

and so $w - v + \frac{1}{4}u = 0$; this gives $v = 0$. From $h(c_1, c_1) = h(c_1, c)$, we then deduce as before that $u = -2$ and $w = \frac{1}{2}$, hence $c_1 = -2a + \frac{1}{2}c$. Thus

$$c_0 = c - c_1 - c_2 = a + \frac{1}{2} b + \frac{1}{4} c.$$

To summarize, we find that necessarily $h(a,c) > 0$; moreover

$$F^3 = 0, \quad F^2 = \mathbb{C}a, \quad F^1 = \mathbb{C}a \oplus \mathbb{C}b, \quad F^0 = \mathbb{C}a \oplus \mathbb{C}b \oplus \mathbb{C}c.$$

We therefore get a Hodge-Lefschetz structure, in which $E_2(H) = \mathbb{C}a$ has weight 4 and type $(2,2)$, and $E_0(H) = \mathbb{C}b$ has weight 2 and type $(1,1)$, and $E_{-2}(H) = \mathbb{C}c$ has weight 0 and type $(0,0)$. The polarization condition is clearly satisfied because

$$(-1)^2 h(a, Y^2 a) = 2h(a,c) > 0.$$

Along the way, we determined the Hodge decomposition of $c = c_0 + c_1 + c_2$ to be

$$c_0 = a + \frac{1}{2} b + \frac{1}{4} c$$
$$c_1 = -2a + \frac{1}{2} c$$
$$c_2 = a - \frac{1}{2} b + \frac{1}{4} c.$$

Inverting these relations, we get

$$\begin{aligned} 4a &= c_0 - c_1 + c_2 \\ b &= c_0 - c_2 \\ c &= c_0 + c_1 + c_2 \end{aligned}$$

Remember these relations, because they will appear again later on.

Outline of the proof. The proof of Theorem 14.1 is going to be a mix of Hodge theory and representation theory. The general idea is the following. For each $\ell \geq 0$, the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has a unique irreducible representation S_ℓ of dimension $\ell+1$. For example, $S_0 = \mathbb{C}$ is the trivial representation; $S_1 = \mathbb{C}^2$ is the standard representation; etc. We saw in Lecture 2 that every finite-dimensional representation V of $\mathfrak{sl}_2(\mathbb{C})$ decomposes into irreducible representations, which means that

$$V \cong \bigoplus_{\ell \in \mathbb{N}} S_\ell \otimes_{\mathbb{C}} W_\ell,$$

where each W_ℓ is a finite-dimensional $\mathbb{C}$-vector space (on which $\mathfrak{sl}_2(\mathbb{C})$ acts trivially). This is a coordinate-free way of saying "$\dim W_\ell$ many copies of S_ℓ". For a more intrinsic description, recall Schur's lemma:

$$\operatorname{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(S_\ell, S_{\ell'}) = \begin{cases} \mathbb{C} \cdot \mathrm{id} & \text{if } \ell = \ell', \\ 0 & \text{if } \ell \neq \ell'. \end{cases}$$

Schurs' lemma implies that $W_\ell \cong \operatorname{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(S_\ell, V)$; we can therefore write the above decomposition in the more natural form

$$V \cong \bigoplus_{\ell \in \mathbb{N}} S_\ell \otimes_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(S_\ell, V), \tag{14.4}$$

where each summand on the right maps to V in the obvious way.

The idea for proving Theorem 14.1 is to show that the decomposition in (14.4) is compatible with the polarized $\mathfrak{sl}_2(\mathbb{C})$-Hodge structure on V. More precisely, we first construct a polarized Hodge-Lefschetz structure (of central weight ℓ) on each S_ℓ. We then show that the vector space $\operatorname{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(S_\ell, V)$ has a polarized Hodge structure of weight $n - \ell$, and that the decomposition in (14.4) holds on the level of polarized $\mathfrak{sl}_2(\mathbb{C})$-Hodge structures. Since the theorem is true for each S_ℓ, it must then also be true for V.

The irreducible representations. We start with the construction for all the irreducible representations S_ℓ.

Example 14.5. On the trivial representation $S_0 = \mathbb{C}$, we can use the pairing $h(x, y) = x\bar{y}$. If we define the filtration by setting $F^1 = 0$ and $F^0 = \mathbb{C}$, we get the Hodge structure $\mathbb{C}(0)$.

Example 14.6. The interesting case is $S_1 = \mathbb{C}^2$, with the standard representation

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The vector $a = (1, 0)$ has weight 1, the vector $b = (0, 1)$ has weight -1, and $Ya = b$. A compatible pairing is given by setting

$$h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

meaning that $h(a, b) = 1$. The filtration $F^1 = \mathbb{C}a$ and $F^0 = \mathbb{C}a \oplus \mathbb{C}b$ certainly satisfies the first two conditions for a polarized $\mathfrak{sl}_2(\mathbb{C})$-Hodge structure. Let us check the third condition. We have

$$e^{-\frac{1}{2}Y}F^1 = \mathbb{C}\left(a - \frac{1}{2}b\right)$$

and since $h(a - \frac{1}{2}b, a - \frac{1}{2}b) = -1 < 0$, we do get a polarized Hodge structure of weight 1 with $S_1^{1,0} = \mathbb{C}(a - \frac{1}{2}b)$ and $S_1^{0,1} = \mathbb{C}(a + \frac{1}{2}b)$. Note that the two vectors $a - \frac{1}{2}b$ and $a + \frac{1}{2}b$ form an orthonormal basis with respect to the inner product induced by the Hodge structure; it follows that the eigenspaces $E_1(H) = \mathbb{C}a$ and $E_{-1}(H) = \mathbb{C}b$ are orthogonal under the inner product.

Example 14.7. Now let us consider S_ℓ for $\ell \in \mathbb{N}$. The best way to do the construction is to say that $S_\ell = \operatorname{Sym}^\ell S_1$ is simply the ℓ-th symmetric product of S_1, and as such, it inherits a polarized $\mathfrak{sl}_2(\mathbb{C})$-Hodge structure of weight ℓ. For the sake of clarity, let me describe the resulting structure in concrete terms. We have

$$S_\ell = \mathbb{C}e_0 \oplus \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_\ell,$$

where $e_0 = a^\ell$, $e_1 = a^{\ell-1}b$, …, $e_\ell = b^\ell$ (as elements of $\operatorname{Sym}^\ell S_1$). One checks that

$$He_j = (\ell - 2j)e_j, \quad Ye_j = (\ell - j)e_{j+1}, \quad Xe_j = je_{j-1},$$

and that the only nontrivial values of the pairing h are

$$h(e_j, e_{\ell-j}) = \frac{1}{\binom{\ell}{j}}.$$

In particular, $h(e_0, e_\ell) = 1$. The filtration F is given by

$$F^p = \mathbb{C}e_0 \oplus \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_{\ell-p},$$

so in particular, $F^{\ell+1} = 0$, $F^\ell = \mathbb{C}e_0$, and $F^0 = S_\ell$. The filtration $e^{-\frac{1}{2}Y}F$ defines a Hodge structure of weight ℓ on S_ℓ, which is just the ℓ-th symmetric product of the Hodge structure of weight 1 on S_1; concretely, we have

$$S_\ell^{p,q} = \mathbb{C}\left(a - \frac{1}{2}b\right)^p\left(a + \frac{1}{2}b\right)^q$$

for $p + q = \ell$, which can of course be expanded in terms of $e_0, \dots, e_\ell$. These vectors are orthogonal with respect to the inner product induced by the Hodge structure, and their square length is $1/\binom{\ell}{p}$. The result for S_1 implies that the different eigenspaces $E_j(H)$ are again orthogonal under the inner product.

Note. In case you are wondering how the formulas for the pairing come about, remember that a hermitian pairing $h\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ induces a hermitian pairing on $\operatorname{Sym}^\ell V$ by the formula

$$h\big(v_1 \cdots v_\ell, w_1 \cdots w_\ell\big) = \frac{1}{\ell!}\sum_{\sigma \in \mathfrak{S}_\ell}\prod_{j=1}^{\ell} h\big(v_j, w_{\sigma(j)}\big),$$

where $\mathfrak{S}_\ell$ is the group of permutations of the set $\{1, \dots, \ell\}$. Similarly, if V has a positive-definite inner product $\langle\,\rangle$, then the induced inner product on $\operatorname{Sym}^\ell V$ is again given by the formula

$$\big\langle v_1 \cdots v_\ell, w_1 \cdots w_\ell\big\rangle = \frac{1}{\ell!}\sum_{\sigma \in \mathfrak{S}_\ell}\prod_{j=1}^{\ell}\big\langle v_j, w_{\sigma(j)}\big\rangle.$$

From this, it is easy to see that if $e_1, \dots, e_n \in V$ are an orthonormal basis, then the vectors $e_1^{a_1} \cdots e_n^{a_n} \in \operatorname{Sym}^\ell V$ with $a_1 + \cdots + a_n = \ell$ are mutually orthogonal, and

$$\|e_1^{a_1} \cdots e_n^{a_n}\|^2 = \frac{a_1! \cdots a_n!}{\ell!}.$$

Hodge structure on the invariant subspace. Now let us return to an arbitrary polarized $\mathfrak{sl}_2(\mathbb{C})$-Hodge structure V of weight n. The first thing to do is to analyze the subspace of $\mathfrak{sl}_2(\mathbb{C})$-invariants; everything else is going to follow from this case.

Proposition 14.8. *Let V be a polarized $\mathfrak{sl}_2(\mathbb{C})$-Hodge structure of weight n. Then*

$$V^{\mathfrak{sl}_2(\mathbb{C})} = \left\{ v \in V \;\middle|\; Hv = Yv = 0 \right\}$$

has a Hodge structure of weight n, polarized by the restriction of h, whose Hodge filtration is $F \cap V^{\mathfrak{sl}_2(\mathbb{C})}$. This Hodge structure is compatible with the polarized Hodge structure on V defined by the filtration $e^{-\frac{1}{2}Y}F$.

The idea of the proof is to carefully analyze the properties of the two operators $Y, H \in \operatorname{End}(V)$. We denote by

$$V = \bigoplus_{p+q=n} V^{p,q}$$

the Hodge structure of weight n with Hodge filtration $e^{-\frac{1}{2}Y}F$; it is polarized by the pairing h. Recall from Lecture 6 that $E = \operatorname{End}(V)$ inherits a Hodge structure of weight 0, with

$$E^{\ell,-\ell} = \left\{ A \in \operatorname{End}(V) \;\middle|\; A(V^{p,q}) \subseteq V^{p+\ell,q-\ell} \text{ for all } p,q \in \mathbb{Z} \right\}.$$

It is polarized by the trace pairing $(A,B) \mapsto \operatorname{tr}(AB^*)$, where $B^* \in \operatorname{End}(V)$ denotes the adjoint of $B \in \operatorname{End}(V)$ with respect to h. Also recall that this is an $\mathbb{R}$-Hodge structure, where the real structure on E is given by

$$E_{\mathbb{R}} = \left\{ A \in \operatorname{End}(V) \;\middle|\; A^* = A \right\}.$$

We may write the Hodge decompositions of $Y, H \in E$ as

$$Y = \sum_{\ell \in \mathbb{Z}} Y_\ell \quad \text{and} \quad H = \sum_{\ell \in \mathbb{Z}} H_\ell.$$

Our assumptions on Y and H can then be expressed as follows.

Lemma 14.9. *We have*

$$Y = Y_{-1} + Y_0 + Y_1 \quad \textit{and} \quad H = Y_{-1} + H_0 - Y_1,$$

and these operators satisfy $Y_{-1}^* = Y_1$, $Y_0^* = Y_0$, and $H_0^* = -H_0$.

Proof. From the fact that $Y^* = Y$, we get $Y_\ell^* = Y_{-\ell}$ for every $\ell \in \mathbb{Z}$. The condition $YF^p \subseteq F^{p-1}$ implies that $Ye^{-\frac{1}{2}Y}F^p \subseteq e^{-\frac{1}{2}Y}F^{p-1}$, which means that $Y_\ell = 0$ for $\ell \leq -2$. But then also $Y_\ell = 0$ for $\ell \geq 2$, and so actually

$$Y = Y_{-1} + Y_0 + Y_1.$$

Similarly, we have $H^* = -H$, hence $H_\ell^* = -H_{-\ell}$ for every $\ell \in \mathbb{Z}$. The condition $HF^p \subseteq F^p$ implies that

$$(H - Y)e^{-\frac{1}{2}Y}F^p = e^{-\frac{1}{2}Y}HF^p \subseteq e^{-\frac{1}{2}Y}F^p,$$

and so $H_\ell - Y_\ell = 0$ for $\ell \leq -1$. In particular, $H_{-1} = Y_{-1}$ and $H_\ell = 0$ for $\ell \leq -2$. This implies $H_\ell = 0$ for $\ell \geq 2$, as well as $H_1 = -H_{-1}^* = -Y_{-1}^* = -Y_1$. □

The relation $[H,Y] = -2Y$ gives us the following additional identity.

Lemma 14.10. *We have $2Y_0 = 2[Y_1, Y_{-1}] + [Y_0, H_0]$.*

Proof. Consider the component of $2Y = [Y,H]$ in the subspace $E^{0,0}$. From the preceding lemma, we get

$$2Y_0 = [Y_1, H_{-1}] + [Y_0, H_0] + [Y_{-1}, H_1] = [Y_1, Y_{-1}] + [Y_0, H_0] - [Y_{-1}, Y_1],$$

which simplifies to the desired identity. □

We can now prove Proposition 14.8.

Proof of Proposition 14.8. On V, we have a polarized Hodge structure of weight n with Hodge filtration $e^{-\frac{1}{2}Y}F$. It is enough to show that $V^{\mathfrak{sl}_2(\mathbb{C})}$ is a sub-Hodge structure; the remaining assertions then follow because $e^{-\frac{1}{2}Y}F$ and F induce the same filtration on $V^{\mathfrak{sl}_2(\mathbb{C})}$ (due to the fact that Y acts trivially).

Now to say that $V^{\mathfrak{sl}_2(\mathbb{C})}$ is a sub-Hodge structure means that whenever we take a vector $v \in V^{\mathfrak{sl}_2(\mathbb{C})}$, and write its Hodge decomposition as

$$v = \sum_p v_p,$$

with $v_p \in V^{p,q}$, then each $v_p \in V^{\mathfrak{sl}_2(\mathbb{C})}$. This is trivially satisfied if $v = 0$. If $v \neq 0$, let $p \in \mathbb{Z}$ be the least integer such that $v_p \neq 0$. It is clearly enough to prove that $v_p \in V^{\mathfrak{sl}_2(\mathbb{C})}$, because we can then repeat the same argument for $v - v_p$. Our goal is therefore to show that $Yv_p = Hv_p = 0$.

From the fact that $Yv = 0$ and the Hodge decomposition, we deduce that

$$Y_{-1}v_p = 0 \quad \text{and} \quad Y_0 v_p + Y_{-1}v_{p+1} = 0.$$

From the fact that $Hv = 0$, we similarly deduce

$$Y_{-1}v_p = 0 \quad \text{and} \quad H_0 v_p + Y_{-1}v_{p+1} = 0.$$

In particular, $Y_0 v_p = H_0 v_p$, which is interesting, because Y_0 and H_0 behave very differently under taking adjoints. We can use this different behavior to show that $h(Y_0 v_p, v_p) = 0$. Since $Y_0^* = Y_0$ and $H_0^* = -H_0$, we have

$$h(Y_0 v_p, v_p) = h(H_0 v_p, v_p) = -h(v_p, H_0 v_p) = -h(v_p, Y_0 v_p) = -h(Y_0 v_p, v_p),$$

and therefore $h(Y_0 v_p, v_p) = 0$. Now we combine this with the identity in Lemma 14.10. Because we already know that $Y_{-1}v_p = 0$, this gives

$$\begin{aligned} 0 = 2h(Y_0 v_p, v_p) &= -2h(Y_{-1}Y_1 v_p, v_p) + h(Y_0 H_0 v_p, v_p) - h(H_0 Y_0 v_p, v_p) \\ &= -2h(Y_1 v_p, Y_1 v_p) + h(H_0 v_p, Y_0 v_p) + h(Y_0 v_p, H_0 v_p) \\ &= -2h(Y_1 v_p, Y_1 v_p) + 2h(Y_0 v_p, Y_0 v_p). \end{aligned}$$

Here we used the fact that $Y_{-1}^* = Y_1$ and $H_0^* = -H_0$, and also $H_0 v_p = Y_0 v_p$. Now $Y_0 v_p \in V^{p,q}$, and since h is a polarization,

$$h(Y_0 v_p, Y_0 v_p) = (-1)^p \|Y_0 v_p\|^2,$$

where $\| \, \|$ is the Hodge norm. Likewise, $Y_1 v_p \in V^{p+1,q-1}$, and so

$$h(Y_1 v_p, Y_1 v_p) = (-1)^{p+1} \|Y_1 v_p\|^2.$$

Putting everything together, we find that

$$0 = \|Y_1 v_p\|^2 + \|Y_0 v_p\|^2,$$

which clearly implies that $Y_1 v_p = 0$ and $Y_0 v_p = 0$. But then also $H_0 v_p = 0$, and so we have proved that $Yv_p = Hv_p = 0$, hence $v_p \in V^{\mathfrak{sl}_2(\mathbb{C})}$. $\square$